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Classical dynamics of an extended charge subjected to a linear force field and Rayleigh–Jeans radiation for a wide class of charge distributions

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Abstract. We study the solutions of the equation of motion for a classical extended charge in the presence of a linear force. In very general conditions, we obtain the qualitative behaviour of the solutions, which for radii greater than a critical value do not present runaways, and display two different terms: one decaying with time and one that oscillates with bounded and not decreasing amplitude. For a slightly more restricted class corresponding essentially to radii not too close to the critical one, but much smaller than the length travelled by the light in a period of the linear force, we show that the non-radiating oscillations do not exist and the solution looks very similar to that of the Abraham-Lorentz model. We also analyse some aspects concerning the interaction of the oscillator with Rayleigh-Jeans radiation and show that its effect is important even upon the phase-space trajectory unless the electromagnetic mass is much smaller than the mechanical one. The results of this paper extend those of a previous paper dealing with a Yukawa charge distribution to a wide class of charge distributions.

1. Introduction

Classical theories of the extended electron have been proposed as alternatives to point theories so as to overcome the well known strange effects of the latter, namely pre-acceleration, runaways and infinite self-energy. (Interesting reviews of this subject can be found in Erber (1961), Coleman (1982), Pearle (1982).) Except for the detailed work by Nodvik (1964), the studies appearing in the literature consider the case of rigid spherically symmetric charge distributions in the non-relativistic approximation. Moreover, the translational and rotational motions are considered separately (see references in Blanco (1987)). Within this context the study of the motion of the extended electron is usually restricted to considering particular charge distributions, such as the uniform sphere (Bohm and Weinstein 1948), the shell (Bohm and Weinstein 1948, Daboul and Jensen 1973a, b, França *et al* 1978, Grandy and Aghazadeh 1982, Levine *et al* 1977, Moniz and Sharp 1977), the Yukawa-type distribution (Blanco *et al* 1986, de la Peña *et al* 1982) and others (Markov 1946). A more general treatment was initiated by the author in the previous paper (Blanco 1987). This work was devoted to analysing the problem of causality of the model for a rigid spherically symmetric charge distribution and in the non-relativistic approximation. In the present paper we intend to go further into the characteristics of this model, with the aid of the analysis of the simplest force field that admits exact solution, namely the linear force. Our aim is to find the behaviour of the solutions with as general a charge distribution as possible.

This work generalises for a wide class of charge distributions most of the results of Blanco *et al* (1986) in which only a Yukawa charge distribution was considered. However, we continue restricting ourselves to rigid spherically symmetric charge distributions and to the non-relativistic regime. Other limitations will be specified in the text.

Our interest is also related to the possibility of justifying the (point-like) Abraham–Lorentz (AL) model as based on a particular expansion starting from the extended model. In other words, we are interested in seeing to what extent the AL model is an approximation to the extended model. Indeed, we shall see that this is guaranteed in certain conditions, the most interesting of which is that the radius must be not too small, and that otherwise different behaviours can appear depending on the specific model of charge distribution we are dealing with. Related to this, the point arises of the perturbative character of the damping, which is discussed in this paper.

Another point of interest is the asymptotic behaviour of the solutions which is related to the existence of runaways and non-radiating solutions.

Following the generalisation of the results obtained in Blanco *et al* (1986), we also present some features of the interaction of the extended charged oscillator with Rayleigh–Jeans (RJ) radiation. The interest in this sort of radiation lies in the fact that the system fulfills a fluctuation–dissipation property (see § 6) which allows for radiative equilibrium for an arbitrary conservative force (not only a linear one), as is shown in Blanco and Pesquera (1986).

The paper is arranged as follows. In § 2 we explain the model whose general solution appears in § 3. Sections 4 and 5 are devoted to a study of the explicit form of the solutions imposing certain weak restrictions on the charge distribution. In § 6 the interaction with the RJ radiation is studied. Finally, the conclusions are discussed in § 7. Two appendices are included to shorten calculations in the main text.

2. The model

The equation of motion for an extended electron has been given in several papers (see, e.g., de la Peña *et al* 1982, França *et al* 1978, Kaup 1966). Here we only give the final expressions.

Let $e\rho(r)$ be the charge density as a function of the distance to the centre of the distribution and let us denote by m_0, m_e and m, the mechanical, electromagnetic and observable masses respectively. The latter fulfil

$$m = m_0 + m_e. \tag{2.1}$$

Introducing

$$m_1 = m_0 - \tfrac{1}{3}m_e = m - \tfrac{4}{3}m_e \tag{2.2}$$

$$\hat{\rho}(\omega) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3\xi\, \rho(\xi) \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\xi}) \qquad c|\boldsymbol{k}| = \omega \tag{2.3}$$

$$\gamma(t) = \frac{32\pi^2 e^2}{3m_1 c^3} \int_0^\infty \omega\hat{\rho}^2(\omega) \sin \omega t \,\mathrm{d}\omega \tag{2.4a}$$

$$= \frac{8\pi e^2}{3m_1} t \int \mathrm{d}^3\xi\, \rho(\xi)\rho(|\boldsymbol{r}+\boldsymbol{\xi}|) \qquad |\boldsymbol{r}| = ct \tag{2.4b}$$

and considering only one-dimensional motion (in the x direction), the equation of motion for the coordinate of the centre of the charge, x, can be written as follows:

$$\ddot{x} = \frac{F}{m_1} - \int_{-\infty}^{t} \mathrm{d}t' \, \gamma(t-t')\ddot{x}(t') \tag{2.5}$$

where F denotes the effect of the external force upon the charge.

The following relations will be useful:

$$\varepsilon \equiv \int_0^\infty \gamma(t)\, \mathrm{d}t = \tfrac{4}{3} m_e / m_1 \tag{2.6}$$

$$\int_0^\infty t\gamma(t)\, \mathrm{d}t = (1+\varepsilon)\tau_0 \tag{2.7}$$

$$m = m_1(1+\varepsilon) \tag{2.8}$$

τ_0 being, as usual, $2e^2/3mc^3$.

Note that if we multiply both terms of (2.7) by m_1 the RHS results in a quantity independent of the charge distribution. This fact will be important later on.

We recall (see Blanco 1987, hereafter referred to as I) the existence of two special radii, namely r_{cr} and r_u, such that

$$m_1(r_{cr}) = 0 \qquad \varepsilon(r_{cr^+}) = +\infty \qquad \varepsilon(r_{cr^-}) = -\infty \tag{2.9a}$$

$$\varepsilon(r_u) = 1 \tag{2.9b}$$

$$r_u > r_{cr}. \tag{2.9c}$$

In (2.5) we shall assume that the force is turned on at a time $t=0$, and that prior to this time the charge is not accelerated. Remember that in I we have shown that for certain radii a non-vanishing acceleration could be a solution without any external force.

The field force in which we are interested is given at any point $\boldsymbol{r}$ by

$$\boldsymbol{F}(\boldsymbol{r}) = -m\omega_0^2 \boldsymbol{r}. \tag{2.10}$$

Its effect upon the charge will be

$$\boldsymbol{F}(x) = \int \boldsymbol{F}(\boldsymbol{r})\rho(|\boldsymbol{r} - x\boldsymbol{i}|)\, \mathrm{d}^3 r = -m\omega_0^2 x\boldsymbol{i} = F\boldsymbol{i} \tag{2.11}$$

where $x\boldsymbol{i}$ denotes the position of the centre of the charge. Consequently, (2.5) now becomes

$$\ddot{x} = -\omega_1^2 x - \int_0^t \mathrm{d}t' \, \gamma(t-t')\ddot{x}(t') \tag{2.12}$$

with

$$\omega_1^2 = \omega_0^2(1+\varepsilon). \tag{2.13}$$

So far we have no restrictions, with the exception of the non-relativistic condition

$$v/c \ll 1. \tag{2.14}$$

In fact, equation (2.5) requires another condition, the explanation of which is outside the subject of the present paper, and can be found in Nodvik (1964). This condition is

$$ar_{e} \ll c^2 \tag{2.15}$$

where a denotes the acceleration and r_e the electron radius.

In addition, in the following analysis we shall be obliged to make stronger restrictions to increase our knowledge of the solutions.

3. General solution

As I have shown in I the solution of (2.12) is unique for each initial condition x_0, v_0.

A straightforward calculation gives

$$x(t) = x_0\chi_1(t) + \dot{x}_0\chi_0(t) \tag{3.1a}$$

$$\dot{x}(t) \equiv v(t) = -x_0\omega_1^2\chi_2(t) + \dot{x}_0\chi_1(t) \tag{3.1b}$$

where the χ_i, $i = 0, 1, 2$, satisfy the following relations:

$$\chi_0(0) = \dot{\chi}_1(0) = \chi_2(0) = 0 \tag{3.2a}$$

$$\dot{\chi}_0(0) = \chi_1(0) = \dot{\chi}_2(0) = 1 \tag{3.2b}$$

$$\chi_0(t) = \chi_2(t) + \int_0^t \gamma(t-s)\chi_2(s)\, \mathrm{d}s \tag{3.3}$$

$$\chi_1(t) = \dot{\chi}_0(t) = 1 - \omega_1^2 \int_0^t \chi_2(s)\, \mathrm{d}s. \tag{3.4}$$

The Laplace transforms of the functions χ_i, $i = 0, 1, 2$, will be useful in the following:

$$\tilde{\chi}_0(p) = (1 + \tilde{\gamma}(p))\tilde{\chi}_2(p) \tag{3.5a}$$

$$\tilde{\chi}_1(p) = p\tilde{\chi}_0(p) \tag{3.5b}$$

$$\tilde{\chi}_2(p) = [\omega_1^2 + p^2(1 + \tilde{\gamma}(p))]^{-1}. \tag{3.5c}$$

4. Runaways and non-radiating oscillations

In this section we introduce new restrictions upon our model, which will allow us to characterise the solutions as far as runaways and non-radiating oscillations are concerned. It will be seen in the analysis below that removing these conditions would demand the study of particular charge distributions. In other words, the analysis seems to indicate that nothing can be said if such conditions are not imposed.

We shall consider in the following that

(A) the memory function is always non-negative,

$$\gamma(t) \geqslant 0 \tag{4.1}$$

(B) $\dot{\gamma}$ exists and is bounded,

(C) the radius is larger than the critical one, i.e.

$$r > r_{cr} \qquad m_1 > 0 \qquad \varepsilon > 0 \tag{4.2}$$

(D) the abscissa of convergence of $\tilde{\gamma}$, σ_c, is strictly negative.

With the exception of (B), which is shown in I to hold in very general and physically reasonable conditions, these assumptions impose what we should think of as important restrictions upon the kind of charge distribution. For instance, (A) is satisfied if ρ has a definite sign (see (2.4*b*)) but not otherwise, in general.

As concerns the radius, for values smaller than r_{cr} it seems (see I) that the system would display runaway behaviour, although this point needs a specific study. Finally, (C) imposes restrictions as to how fast the charge density decays for long distances. An as yet unknown relation exists between the decay rates of ρ and γ, and (C) imposes that γ decreases faster than an exponential, i.e.

$$\exists\alpha/\gamma(t) \underset{t\to\infty}{<} \text{constant}\times e^{-\alpha t}. \tag{4.3}$$

We think that the study of systems not fulfilling (A), (C) and (D) would be convenient, for new behaviour different from the ones we shall obtain in this paper could appear.

Returning to our analysis, if we want to find the behaviour of the functions χ_i we first calculate the zeros of

$$\Pi(p) \equiv \omega_1^2 + p^2[1+\tilde{\gamma}(p)] \tag{4.4}$$

which are the poles of χ_i, $i = 0, 1, 2$.

With the conditions stated above we shall show the following.

(i) There are no zeros with positive real part.

(ii) Zeros having their real part larger than the abscissa of convergence of $\tilde{\gamma}$ are isolated points.

(iii) The number of zeros lying in any region of the complex plane between two vertical lines, $a_1 \leqslant \operatorname{Re} p \leqslant a_2$, $a_i > \sigma_c$, is finite. In particular, the number of zeros lying over the imaginary axis is finite.

(iv) The zeros with null real part have multiplicity one.

Before proving these properties we shall expose their consequences.

From (iii), for any $\eta < 0$ there are a finite number of zeros of $\Pi(p)$ with real part between η and 0. Consequently, we may choose $\eta_0 < 0$ such that the zeros lie either on the imaginary axis or to the left of the line $C_1 \equiv \operatorname{Re} p = \eta_0$. Let C_2 be any other vertical line with positive real part. By using the inverse Laplace transform we have

$$\chi_a(t) = \frac{1}{2\pi i}\int_{C_{2\uparrow}} dp\, e^{pt}\tilde{\chi}_\alpha(p) \qquad t>0;\ \alpha = 0, 1, 2. \tag{4.5}$$

Now, since $\tilde{\chi}_\alpha(p) \to_{p\to\infty} 0$ it is straightforward to pass from the integral along C_2 to the integral along C_1, taking into account the zeros of $\Pi(p)$ between both lines. These zeros are all on the imaginary axis, by construction. These, according to (iii) and (iv), are simple and finite in number.

Consequently we may write

$$\chi_\alpha(t) = \frac{1}{2\pi i}\int_{C_{1\uparrow}} dp\, e^{pt}\tilde{\chi}_\alpha(p) + \sum_n K_\alpha(i\mu_n)\exp(i\mu_n t) \qquad \alpha = 0, 1, 2 \tag{4.6}$$

where $i\mu_n$ denote the imaginary zeros of $\Pi(p)$ and $K_\alpha(i\mu_n)$ is the residue of $\tilde{\chi}_\alpha$ at the point $p = i\mu_n$. The first integral of (4.6) can also be written

$$\exp(-|\operatorname{Re} p|t)\frac{1}{2\pi}\int_{-\infty}^{\infty} dp_2\exp(ip_2 t)\tilde{\chi}_\alpha(\operatorname{Re} p + ip_2). \tag{4.7}$$

Thus this integral decreases to zero as long as the time elapses.

Now, we have obtained that the solutions are bounded in time: no runaway solutions appear. Nevertheless non-radiating solutions can exist corresponding to the second term in (4.6). These sorts of solutions have been known since the beginning of the century. They were first encountered by Sommerfeld (see Erber 1961). In our case the existence of these states is only approximated due to the non-relativistic treatment. However, it is important to draw attention to the fact that, in a relativistic treatment, these solutions are still possible (Devaney and Wolf 1973, Fargue 1981).

4.1. *Proof of properties (i)-(iv)*

Let us consider equation (4.4) separated into its real and imaginary parts:

$$\omega_1^2 \operatorname{Re}(p^{-2}) + 1 + \operatorname{Re} \tilde{\gamma} = 0 \tag{4.8a}$$

$$\omega_1^2 \operatorname{Im}(p^{-2}) + \operatorname{Im} \tilde{\gamma} = 0. \tag{4.8b}$$

Let us write

$$p = \lambda + \mathrm{i}\mu \tag{4.9}$$

with $\lambda, \mu \in \mathbb{R}$.

To prove (i) let us suppose $\lambda > 0$. By using (2.4a), equation (4.8b) may be written as

$$\frac{-\omega_1^2 2\mu\lambda}{(\lambda^2-\mu^2)^2+4\mu^2\lambda^2} - \frac{32\pi^2 e^2}{3m_1c^3} 2\mu\lambda \int_0^\infty \mathrm{d}\omega\, \hat{\rho}^2(\omega) \frac{\omega^2}{(\lambda^2-\mu^2+\omega^2)^2+4\mu^2\lambda^2} = 0 \tag{4.10}$$

whence $\mu = 0$ due to the fact that, by hypothesis, $\lambda > 0$ and $m_1 > 0$. Consequently, equation (4.8a) results in

$$\frac{\omega_1^2}{\lambda^2} + 1 + \frac{32\pi^2 e^2 \lambda^2}{3m_1c^3} \int_0^\infty \mathrm{d}\omega\, \hat{\rho}^2(\omega) \frac{\omega^2}{(\lambda^2+\omega^2)^2} = 0 \tag{4.11}$$

which has no solutions again due to $m_1 > 0$. This proves (i). Note that this is a consequence of condition (C) only.

(ii) is a consequence of the fact that, for $\operatorname{Re} p > \sigma_c$, $\Pi(p)$ is an analytic and non-identically vanishing function.

To prove (iii), condition (B) allows us to write

$$\left|\frac{\tilde{\gamma}(\lambda+\mathrm{i}\mu)}{\mu}\right| = \left|\frac{(\lambda+\mathrm{i}\mu)\tilde{\gamma}(\lambda+\mathrm{i}\mu)}{\mu}\right| \leqslant \frac{\int_0^\infty |\dot{\gamma}(t)|\, \mathrm{e}^{-\lambda t}\, \mathrm{d}t}{|\mu|}. \tag{4.12}$$

Now,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \int_0^\infty |\dot{\gamma}(t)|\, \mathrm{e}^{-\lambda t}\, \mathrm{d}t = -\int_0^\infty t|\dot{\gamma}(t)|\, \mathrm{e}^{-\lambda t}\, \mathrm{d}t < 0 \tag{4.13}$$

whence the Laplace transform of $|\dot{\gamma}(t)|$ is strictly decreasing:

$$a_1 < a_2 \Rightarrow \widetilde{|\dot{\gamma}|}(a_1) > \widetilde{|\dot{\gamma}|}(a_2). \tag{4.14}$$

Consequently, for any λ in $[a_1, a_2]$, (4.12) and (4.14) give

$$|\tilde{\gamma}(\lambda+\mathrm{i}\mu)| \leqslant \frac{\int_0^\infty |\dot{\gamma}(t)|\, \mathrm{e}^{-a_1 t}}{|\mu|} \tag{4.15}$$

and therefore it is possible to find μ_0 fulfilling

$$\forall \mu > \mu_0 \qquad |\omega_1^2/\mu^2| < \tfrac{1}{4} \qquad |\tilde{\gamma}(a_1 + \mathrm{i}\mu)| < \tfrac{1}{4}. \tag{4.16}$$

Let us consider now the region of the complex plane defined by

$$D \equiv \{p \in \mathbb{C}/a_1 \leqslant \mathrm{Re}\, p \leqslant a_2;\ |\mathrm{Im}\, p| > \mu_0\}. \tag{4.17}$$

In this region, from (4.12), (4.14) and (4.16) it results that

$$|\omega_1^2/p^2 + \tilde{\gamma}(p)| \leqslant \omega_1^2/\mu^2 + |\tilde{\gamma}(p)| < \tfrac{1}{2} \tag{4.18}$$

and then $\Pi(p)$ cannot have zeros in D. Consequently, the zeros of $\Pi(p)$ between $\mathrm{Im}\, p = a_1$ and $\mathrm{Im}\, p = a_2$ lie in a compact domain and, as they are isolated, their number is necessarily finite. This proves (iii).

Finally, to prove (iv) we shall show that

$$\Pi(\mathrm{i}\mu_0) = 0 \Rightarrow \Pi'(\mathrm{i}\mu_0) \neq 0. \tag{4.19}$$

We first prove that $\rho'(\mu_0)$ is finite. To see this, we write (2.3), after a little algebra, as

$$\hat{\rho}(\omega) = \left(\frac{2}{\pi}\right)^{1/2} \frac{c}{\omega} \int_0^\infty \mathrm{d}\xi\, \xi\rho(\xi) \sin(\omega\xi/c). \tag{4.20}$$

Differentiating, we immediately obtain

$$\hat{\rho}'(\omega) = -\frac{1}{\omega}\hat{\rho}(\omega) + \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{\omega} \int_0^\infty \mathrm{d}\xi\, \xi^2\rho(\xi) \cos(\omega\xi/c) \tag{4.21}$$

whence it results that

$$|\hat{\rho}'(\mu_0)| \leqslant \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{\mu_0} \int_0^\infty \mathrm{d}\xi\, \xi^2\rho(\xi) = \frac{1}{\mu_0}\left(\frac{2}{\pi}\right)^{1/2}$$

which is the desired result.

Secondly, from (2.4*a*) a straightforward calculation yields

$$\tilde{\gamma}(\mathrm{i}\mu_0) = \frac{32\pi^2 e^2}{3m_1c^3}\left(-\tfrac{1}{2}\mathrm{i}\mu_0\pi\hat{\rho}^2(\mu_0) + \tfrac{1}{2}\mathrm{VP}\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\omega}{\omega - \mu_0}\hat{\rho}^2(|\omega|)\right) \tag{4.22}$$

(where VP denotes the principal value).

The conditions for $\mathrm{i}\mu_0$ to be a zero of $\Pi(p)$ can now be written, separating the real and imaginary parts of $\Pi(p)/p^2$, as

$$\hat{\rho}(\mu_0) = 0 \tag{4.23a}$$

$$\omega_1^2/\mu_0^2 - 1 = \frac{16\pi^2 e^2}{3m_1c^3}\mathrm{VP}\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\omega}{\omega - \mu_0}\hat{\rho}^2(|\omega|). \tag{4.23b}$$

Note that $\mu_0 = 0$ (i.e. $p = 0$) is not a zero of $\Pi(p)$. Let us now calculate $\Pi'(\mathrm{i}\mu_0)$:

$$\Pi'(\mathrm{i}(\mu_0) = 2\mathrm{i}\mu_0(1 + \tilde{\gamma}(\mathrm{i}\mu_0)) - \mu_0^2\tilde{\gamma}'(\mathrm{i}\mu_0) \tag{4.24}$$

and using (4.4) we obtain

$$\Pi'(\mathrm{i}\mu_0) = -2\omega_1^2/\mathrm{i}\mu_0 - \mu_0^2\tilde{\gamma}'(\mathrm{i}\mu_0) \tag{4.25}$$

whence we obtain

$$\mathrm{Re}[\Pi'(i\mu_0)] = -\mu_0^2 \frac{\mathrm{d}}{\mathrm{d}\mu_0} \mathrm{Im}\ \tilde{\gamma}(\mathrm{i}\mu_0) \tag{4.26a}$$

$$\mathrm{Im}[\Pi'(\mathrm{i}\mu_0)] = \frac{2\omega_1^2}{\mu_0} + \mu_0^2 \frac{\mathrm{d}}{\mathrm{d}\mu_0} \mathrm{Re}\ \tilde{\gamma}(\mathrm{i}\mu_0). \tag{4.26b}$$

From (4.22), (4.23*a*) and the existence of $\hat{\rho}'(\mu_0)$ we readily obtain

$$\operatorname{Re} \Pi'(\mathrm{i}\mu_0) = 0. \tag{4.27}$$

For $\operatorname{Im}(\Pi'(\mathrm{i}\mu_0))$, we show in appendix 1 that

$$\frac{\mathrm{d}}{\mathrm{d}\mu_0} \operatorname{Re} \tilde{\gamma}(\mathrm{i}\mu_0) > 0 \tag{4.28}$$

whence we finally obtain

$$\operatorname{Im}[\Pi'(\mathrm{i}\mu_0)] > 0 \tag{4.29}$$

which proves (iv).

5. Behaviour of the solutions for a localised and sufficiently big charge

A further step in the study of the solutions can only be given if more restrictions are imposed on the model. Indeed, the analysis developed in this section will show the impossibility of predicting the kind of behaviour resulting in this section if the new restrictions defined below did not hold.

We shall consider now the following additional conditions:

(E) the charge distribution has bounded support, i.e.

$$\rho(r) = 0 \qquad \text{for } r > r_\mathrm{e} = c\tau_\mathrm{e} \tag{5.1a}$$

(F) $$\omega_0 \tau_\mathrm{e} \ll 1 \tag{5.1b}$$

(G) $$0 < \varepsilon < 0.5. \tag{5.1c}$$

Some comments on these conditions are necessary.

Condition (E) may be considered as perfectly reasonable, although we might also admit distributions with non-compact support and having the most important contribution within a region of radius r_e. It will be seen, indeed, that in this case a similar behaviour of the solutions is to be expected. Nevertheless, the analysis would become much more complicated.

Condition (F) says that the period T associated with the oscillator is much greater than the time light takes to cross the charge. It is clear that for an electron it constitutes a very small restriction.

As concerns condition (G), the analysis of this section shows it to be necessary in order to predict the behaviour displayed with the other three conditions. One can easily see that, for $\varepsilon = 1$, such behaviour cannot yet be guaranteed. On the other hand, condition (C) implies, according to the results of I, that the homogeneous equation only has a trivial solution, and then we shall always have $\ddot{x} = 0$ before the force is turned on.

Before passing to the analysis of the solutions a consequence of (5.1*a*, *b*, *c*) can be obtained.

In equation (2.4*b*) one readily sees that

$$t > 2\tau_\mathrm{e} \Rightarrow \gamma(t) = 0 \tag{5.2}$$

whence (2.6) and (2.7) give

$$(1+\varepsilon)\tau_0 = \int_0^{2\tau_\mathrm{e}} t\gamma(t)\, \mathrm{d}t \leqslant 2\tau_\mathrm{e} \int_0^{2\tau_\mathrm{e}} \gamma(t)\, \mathrm{d}t = 2\tau_\mathrm{e}\varepsilon \tag{5.3}$$

i.e.

$$\tau_0/\tau_e \leq 2\varepsilon/(1+\varepsilon) < 1 \tag{5.4}$$

where we have made use of (5.1*c*). Consequently, also

$$\omega_0\tau_0 \ll 1. \tag{5.5}$$

This relation is usually considered when studying point models of an electron due to the pre-acceleration phenomenon taking place in a period of time of order τ_0. Equation (5.5) represents the necessity for this time interval to be much smaller than the time characteristic of the oscillator (or, in general, of the external force field).

The relation τ_0/τ_e for a Yukawa-type distribution (Blanco *et al* 1986) is exactly $2\varepsilon/(1+\varepsilon)$. Then, although this charge density does not satisfy (5.1*a*), it is not expected that an upper bound for τ_0/τ_e smaller than the one expressed in (5.4) might be found.

In order to study the behaviour of the solution we consider two steps: first, we analyse the situation of the zeros of $\Pi(p)$ in the complex plane, and then look for a more explicit expression of the solutions. We also include a comparison with the Abraham-Lorentz model.

5.1. Zeros of $\Pi(p)$

The first point here is to show that no non-radiating solutions exist, i.e. there are no zeros over the imaginary axis. To see this we first show that conditions (E) and (F) imply that non-radiating solutions exist only if ε is negative or, if positive, is larger than or very close to 1; in other words, if the electron radius is smaller than or very close to r_u (see equations (2.9*b*, *c*)).

To see it let us suppose that $0<\varepsilon<1$ (condition (G)) is not considered and write equations (4.23*a*, *b*), taking (4.22) into account as

$$\omega_1^2/\mu_0^2 = 1 + \mathrm{Re}\ \tilde{\gamma}(\mathrm{i}\mu_0) \tag{5.6a}$$

$$\mathrm{Im}\ \tilde{\gamma}(\mathrm{i}\mu_0) \equiv -\int_0^\infty \gamma(t)\sin\mu_0 t\ \mathrm{d}t = 0. \tag{5.6b}$$

Now, from equation (2.4*b*) we obtain that $\gamma(t)=0$ for $t>2\tau_e$. Accordingly (5.6*b*) can be written

$$\int_0^{2\tau_e} \gamma(t)\sin\mu_0 t\ \mathrm{d}t = 0. \tag{5.7}$$

However, for $\mu_0 \leq \pi/2\tau_e$, the integrand in (5.7) is non-negative and non-identically zero, and the integral cannot vanish. Consequently, equation (5.7) means that

$$\mu_0 > \pi/2\tau_e > 1/\tau_e. \tag{5.8}$$

Furthermore, it is trivial to see that

$$|\mathrm{Re}\ \tilde{\gamma}(\mathrm{i}\mu_0)| \leq |\tilde{\gamma}(\mathrm{i}\mu_0)| \leq \varepsilon \tag{5.9}$$

and consequently

$$1-\varepsilon < |1+\mathrm{Re}\ \tilde{\gamma}(\mathrm{i}\mu_0)| \tag{5.10}$$

whence, from (2.13), (5.6*a*) and (5.8) we obtain

$$\varepsilon > \frac{1-\omega_0^2\tau_e^2}{1+\omega_0^2\tau_e^2} = 1 - \mathrm{O}(\omega_0^2\tau_e^2) \tag{5.11}$$

as we required.

If condition (G) is now taken into account, it is obvious that it is in contradiction with (5.11). Consequently, no zeros of $\Pi(p)$ lie on the imaginary axis and the oscillatory terms in (4.6) do not exist.

Once this has been proven, we shall show that the zeros are distributed into two groups: two of them are very close to the ones corresponding to the undamped oscillator ($\varepsilon \to 0$), i.e. $p_\pm \simeq \pm i\omega_0$, and the others are located to the left of the vertical line $\text{Re}\, p = -1/4\tau_e$, and outside the region

$$D_1 = \{p \in \mathbb{C} / |\text{Im}\, p| < \pi/4\tau_e\}.$$

Let us first consider the possible zeros in D_1.

The following relations hold:

$$\text{Re}\, \tilde{\gamma}(p) = \int_0^{2\tau_e} \gamma(t) \exp(|\text{Re}\, p| t) \cos(\text{Im}\, pt)\, dt > 0 \tag{5.12}$$

$$\Rightarrow 1 + \text{Re}\, \tilde{\gamma} > 1 \tag{5.13}$$

$$\Rightarrow |1 + \tilde{\gamma}| > 1 \tag{5.14}$$

$$\Rightarrow |\omega_1^2/p^2| = |1 + \tilde{\gamma}(p)| > 1 \tag{5.15}$$

$$\Rightarrow |p| < \omega_0(1+\varepsilon)^{1/2}. \tag{5.16}$$

For this value of p we may expand $\tilde{\gamma}$ in a power series of $\omega_e\tau_0$ and retain the first terms. To see this let us write

$$\tilde{\gamma}(p) = \int_0^{2\tau_e} \gamma(t)\, e^{-pt}\, dt = \sum_{k=0}^{\infty} \frac{(-1)^k p^k}{k!} \int_0^{2\tau_e} \gamma(t) t^k\, dt$$

$$= \varepsilon \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (2p\tau_e)^k T_k \tag{5.17}$$

with

$$T_k = \frac{1}{\varepsilon(2\tau_e)^k} \int_0^{2\tau_e} \gamma(t) t^k\, dt. \tag{5.18}$$

Now

$$0 \leqslant T_{k+1} \leqslant T_k \leqslant 1 \qquad \forall k \tag{5.19}$$

and then the series in (5.17) converges better than an exponential. For $|p| < \omega_0(1+\varepsilon)^{1/2}$, we may write

$$\tilde{\gamma}(p) = [\varepsilon - p\tau_0(1+\varepsilon) + \varepsilon O((\omega_0\tau_e)^2)]. \tag{5.20}$$

Going now to equation (4.4) we obtain

$$0 = \omega_0^2(1+\varepsilon) + p^2[(1+\varepsilon)(1-p\tau_0) + \varepsilon O((\omega_0\tau_e)^2)] \tag{5.21}$$

the solution of which may be obtained in terms of an expansion in $\omega_0\tau_e$. We obtain the following two solutions:

$$p_\pm = \pm i\omega_0 - \tfrac{1}{2}\omega_0^2\tau_0 + \omega_0 O((\omega_0\tau_e)^2) \tag{5.22}$$

which correspond to the undamped oscillator plus terms of order $\omega_0\tau_e$. Note that the second term of the expansion of $p_\pm$ is independent of the particular charge distribution we are considering. This is due to the fact that the $k=1$ term of the expansion (5.17) is distribution independent, because of relation (2.7), i.e.

$$T_1 = \frac{1+\varepsilon}{\varepsilon}\frac{\tau_0}{2\tau_e}. \tag{5.23}$$

Now, we are going to consider the region of the complex plane

$$D_2 = \{p \in \mathbb{C}/|\mathrm{Re}\, p| < 1/4\tau_e, |\mathrm{Im}\, p| > \pi/4\tau_e\}. \tag{5.24}$$

On the one hand, we may write

$$|\mathrm{Re}\, \tilde{\gamma}| \leqslant \int_0^{2\tau_e} \gamma(t) \exp(t/4\tau_e)\, \mathrm{d}t < \varepsilon\sqrt{e} < 0.9 \tag{5.25}$$

whence we obtain

$$|1+\tilde{\gamma}| > |1-\varepsilon\sqrt{e}| > 0.1 \tag{5.26}$$

and, on the other hand,

$$|\omega_1^2/p^2| < \omega_0^2(1+\varepsilon)\tau_e^2 16/\pi^2 < 3\omega_0^2\tau_e^2. \tag{5.27}$$

Because of condition (5.1b), relations (5.25) and (5.27) cannot be fulfilled by the same p, whence we deduce that there are no zeros of $\Pi(p)$ lying on D_2.

Consequently we have found the zeros of $\Pi(p)$ are located within the region $\{p \in \mathbb{C}/\mathrm{Re}\, p \langle -1/4\tau_e, |\mathrm{Im}\, p| \rangle \pi/4\tau_e\}$, i.e. too far from the coordinate axis, except two of them that lie very close to the zeros of the undamped oscillator, and are expressed by (5.22).

5.2. *Behaviour of the solutions*

In this section we will show that the behaviour of solutions is mainly determined by the zeros $p_\pm$. We shall see indeed that the functions $\chi_i(t)$ can be explicitly obtained up to terms of order $\omega_0\tau_e$.

Let us consider two vertical lines in the complex plane C_1 having all zeros of $\Pi(p)$ on its left, and C_2 between $p_\pm$ and the other zeros, the latter defined as $\mathrm{Re}\, p = -1/5\tau_e$.

By using the inverse Laplace transform, expression (4.6) may be written

$$\chi_i(t) = \frac{1}{2\pi \mathrm{i}}\int_{C_{1\uparrow}} \mathrm{d}z\, \mathrm{e}^{zt}\tilde{\chi}_i(z) = \frac{1}{2\pi \mathrm{i}}\int_{C_{2\uparrow}} \mathrm{d}z\, \mathrm{e}^{zt}\tilde{\chi}_i(z) + K_i(p_+)\, \mathrm{e}^{p_+ t} + K_i(p_-)\, \mathrm{e}^{p_- t} \qquad t>0 \tag{5.28}$$

$K_i(p_\pm)$ being the residue of $\tilde{\chi}_i$ in $p_\pm$ respectively. Let us denote by $K_i(t)$ the two last members of the RHS in (5.28), $R_i(t)$ the integral over C_2 and

$$p_\pm = \omega_0(\pm \mathrm{i}\nu - \xi) \tag{5.29}$$

where, by virtue of (5.24),

$$\nu = 1 + \mathrm{O}(\omega_0^2\tau_e^2) \tag{5.30a}$$

$$\xi = \tfrac{1}{2}\omega_0\tau_0 + \mathrm{O}(\omega_0^2\tau_e^2). \tag{5.30b}$$

Let us also write

$$\tilde{\chi}_i(p) = \varphi_i(p)\tilde{\chi}_2(p) \tag{5.31}$$

with

$$\varphi_0(p) = 1 + \tilde{\gamma}(p) \tag{5.32a}$$

$$\varphi_1(p) = p(1 + \tilde{\gamma}(p)) \tag{5.32b}$$

$$\varphi_2(p) = 1. \tag{5.32c}$$

Then

$$K_i(p_+) = \lim_{p \to p_+} \chi_i(p)(p - p_+) = \varphi_i(p_+) K_2(p_+) \tag{5.33}$$

and

$$K_i(t) = 2\ \mathrm{Re}[\varphi_i(p_+) K_2(p_+) \exp(\mathrm{i}\omega_0 \nu t)] \exp(-\omega_0 \xi t). \tag{5.34}$$

Let us then calculate $K_2(p_+)$. Making use of the l'Hôpital rule we obtain

$$K_2(p_+) \equiv \lim_{p \to p_+} \frac{p - p_+}{\omega_1^2 + p^2[1 + \tilde{\gamma}(p)]} = \frac{p_+}{-2\omega_1^2 + p_+^3 \tilde{\gamma}'(p_+)}. \tag{5.35}$$

As we are interested in an $\omega_0 \tau_e$ expansion, note that

$$\frac{|p_+^3 \tilde{\gamma}'(p_+)|}{\omega_1^2} \simeq \frac{\omega_0^3 |\int_0^{2\tau_e} \mathrm{d}t\, \gamma(t) t\, \mathrm{e}^{-p_+ t}|}{\omega_0^2 (1 + \varepsilon)}$$

$$< \frac{\omega_0}{1 + \varepsilon} 2\tau_e \varepsilon \exp(\xi \omega_0 2\tau_e) \sim 2\omega_0 \tau_e \ll 1 \tag{5.36}$$

and then

$$K_2(p_+) = \frac{p_+}{-2\omega_1^2} + \mathrm{O}(\omega_0 \tau_e) = \frac{1}{2\mathrm{i}\omega_0(1 + \varepsilon)} + \mathrm{O}(\omega_0 \tau_e). \tag{5.37}$$

Finally, by using (5.1*a*), (5.25), (5.32*a*, *b*, *c*) and (5.34) simple algebra yields

$$K_0(t) = (1/\omega_0)[\sin(\omega_0 \nu t) + \mathrm{O}(\omega_0 \tau_e)] \exp(-\omega_0 \xi t) \tag{5.38a}$$

$$K_1(t) = [\cos(\omega_0 \nu t) + \mathrm{O}(\omega_0 \tau_e)] \exp(-\omega_0 \xi t) \tag{5.38b}$$

$$K_2(t) = \frac{1}{\omega_0(1 + \varepsilon)} [\sin(\omega_0 \nu t) + \mathrm{O}(\omega_0 \tau_e)] \exp(-\omega_0 \xi t). \tag{5.38c}$$

Now we must investigate the contribution of $R_i(t)$ in (5.28)

$$R_i(t) = \frac{1}{2\pi \mathrm{i}} \int_{C_{2t}} \mathrm{d}z\, \mathrm{e}^{zt} \frac{\varphi_i(z)}{\omega_1^2 + z^2[1 + \tilde{\gamma}(z)]} \qquad t > 0;\ i = 0, 1, 2. \tag{5.39}$$

It will be useful to consider these expressions when $\tilde{\gamma}$ is replaced by ε:

$$R_i^c(t) = \frac{1}{2\pi \mathrm{i}} \int_{C_{2t}} \mathrm{d}z\, \mathrm{e}^{zt} \frac{\varphi_i^c(z)}{(\omega_0^2 + z^2)(1 + \varepsilon)} \qquad t > 0 \tag{5.40}$$

with

$$\varphi_0^c(z) = 1 + \varepsilon \tag{5.41a}$$

$$\varphi_1^c(z) = z(1 + \varepsilon) \tag{5.41b}$$

$$\varphi_2^c(z) = 1. \tag{5.41c}$$

The interesting point is that $R_i^c(t)=0$ $(t>0)$. This can be seen closing the integration contour to the left of C_2 and taking into account that the only poles of the integrand of $R_i^c(t)$ are on the imaginary axis. Consequently we may write

$$R_i(t)=\frac{1}{2\pi i}\int_{C_{2\uparrow}} dz\, e^{zt}\left(\frac{\varphi_i(z)}{\omega_1^2+z^2(1+\tilde\gamma)}-\frac{\varphi_i^c(z)}{(\omega_0^2+z^2)(1+\varepsilon)}\right)$$
$$=\frac{1}{2\pi i}\int_{C_{2\uparrow}} dz\, e^{zt}\frac{\Phi_i(z)}{[\omega_0^2(1+\varepsilon)+z^2(1+\tilde\gamma)](\omega_0^2+z^2)} \qquad t>0 \quad (5.42)$$

where now

$$\Phi_0=\omega_0^2(\tilde\gamma-\varepsilon) \qquad (5.43a)$$
$$\Phi_1=\omega_0^2 z(\tilde\gamma-\varepsilon) \qquad (5.43b)$$
$$\Phi_2=\frac{z^2(\varepsilon-\tilde\gamma)}{1+\varepsilon}. \qquad (5.43c)$$

Equations (5.42) for $i=0,1,2$ are analysed in appendix 2 where the following relations are obtained:

$$\omega_0|R_0(t)|<300(\omega_0\tau_e)^3\exp(-t/5\tau_e) \qquad (5.44a)$$
$$|R_1(t)|<100(\omega_0\tau_e)^2\exp(-t/5\tau_e) \qquad (5.44b)$$
$$\omega_0(1+\varepsilon)|R_2(t)|<25(\omega_0\tau_e)\exp(-t/5\tau_e). \qquad (5.44c)$$

If we compare these expressions with $\omega_0 K_0$ (5.38a), K_1 (5.38b) and $\omega_0(1+\varepsilon)K_2$ (5.38c), respectively, it is clear that the latter are the predominant ones. Indeed the R_i are not only of order $\omega_0\tau_e \ll 1$ with respect to the K_i, but they even decrease with time much faster. The general solution in phase space is written by taking (5.28), (5.38a, b, c) and (5.44a, b, c) to (3.1a, b):

$$x(t)=[x_0\cos(\omega_0\nu t)+(\dot x_0/\omega_0)\sin(\omega_0\nu t)+O(\omega_0\tau_e)]\exp(-\omega_0\xi t)+O(\omega_0\tau_e\exp(-t/5\tau_e)) \qquad (5.45a)$$

$$\dot x(t)=[-x_0\omega_0\sin(\omega_0\nu t)+\dot x_0\cos(\omega_0\nu t)+O(\omega_0\tau_e)]\exp(-\omega_0\xi t)+O(\omega_0\tau_e\exp(-t/5\tau_e)). \qquad (5.45b)$$

From (5.30a) one readily deduces that the motion is approximately that of the undamped oscillator up to times of order

$$\tau_r^{(0)}\sim\frac{1}{\omega_0^2\tau_0}\gg T \qquad (5.46)$$

i.e. during many periods. This behaviour shows that, for models satisfying conditions (A)-(G) the effect of the damping term in the equation of motion is essentially perturbative with respect to the linear force. We want to emphasise the fact that nothing can be said in general if those conditions are not satisfied.

Obviously, in (5.45a, b) we see that the energy decreases with time at a rate

$$dE/dt\simeq-2\omega_0\xi E \qquad (5.47)$$

which is much slower than the oscillations. Moreover, we obtain that the system is non-relativistic provided the initial conditions are non-relativistic.

Finally, as concerns condition (2.15), a similar analysis for $\dot\chi_2$ (which appears in the expression of the acceleration) again shows that appropriate initial values of x_0 and $\dot x_0$ guarantee that (2.15) holds at all times.

5.3. Comparison with the Abraham–Lorentz (AL) model

The general solution of the AL model for $t>0$ can be written

$$x_{\rm AL}=x_0\,{\rm e}^{-\mu t}[\cos\bar\omega t+(\mu/\bar\omega)\sin\bar\omega t]+(\dot x_0/\bar\omega)\,{\rm e}^{-\mu t}\sin\bar\omega t \tag{5.48a}$$

$$\dot x_{\rm AL}=\dot x_0\,{\rm e}^{-\mu t}[\cos\bar\omega t-(\mu/\bar\omega)\sin\bar\omega t]-(x_0/\bar\omega)\,{\rm e}^{-\mu t}(\mu^2+\bar\omega^2)\sin\bar\omega t \tag{5.48b}$$

where μ and $\bar\omega$ can be expressed in powers of $\omega_0\tau_0$:

$$\mu=\tfrac{1}{2}\omega_0^2\tau_0-\omega_0^4\tau_0^3+\omega_0{\rm O}(\omega_0^5\tau_0^5) \tag{5.49a}$$

$$\bar\omega=\omega_0-\tfrac{5}{8}\omega_0^3\tau_0^2+\omega_0{\rm O}(\omega_0^4\tau_0^4). \tag{5.49b}$$

Comparing with (5.45a, b) we see that the AL model solution only accounts for the predominant and slowly oscillating terms of the extended model solution, whereas the last terms in (5.45a, b) do not appear in (5.48a, b). Moreover, as concerns the common terms, we see that they are equal in the first order of $\omega_0\tau_0$ for all parameters appearing in both expressions, i.e. the frequency, $\omega_0\nu$ and $\bar\omega$, the decaying rate, $\omega_0\xi$ and μ, and the coefficients. In fact, the first term of the expansion of μ, (5.49a), coincides with the second term of the expansion of $p_\pm$, (5.22), i.e. the first term of the expansion of $-\omega_0\xi$ (see (5.30b)).

Recall that such a term is the same for all charge distributions, as we said in the comments following (5.22). This is why the decaying rate also coincides in first order for both electron models.

However, all the parameters differ in the second term of their respective expansions in $\omega_0\tau_0$. This is because such terms for the extended model in general depend on each particular charge distribution.

There is another difference between the two models. If one considers the successive derivatives of χ_2 (or $\ddot x(t)$), we find that the rapidly decreasing terms have a contribution which grows as $\tau_{\rm e}^{-n}$ for the nth derivative of χ_2. This is essentially obtained when deriving the exponential $\exp(-t/5\tau_{\rm e})$. Obviously this exponential remains and such behaviour occurs for very short times. Consequently, in the first instance, these rapidly decaying terms are the predominant ones in the case of all derivatives of χ_2. This may be confirmed in the case of the Yukawa distribution, as is seen in Blanco *et al* (1986).

6. Oscillator in the presence of a Rayleigh–Jeans (RJ) radiation

Most of the calculations appearing in Blanco *et al* (1986) for this case are made in fact for a general charge distribution with the exception of the final results, which are valid only for the Yukawa case. This is why in this section we shall explicitly use the general expressions obtained in that paper, to which we refer the reader for the radiation model and the notation.

We only recall that the RJ radiation is modelled as a zero-mean Gaussian stochastic process whose spectrum is given by

$$S(\omega)=4\omega^2k_{\rm B}T/3c^3 \tag{6.1}$$

T being the absolute temperature.

In this context, the solution of the equation of motion

$$\ddot x=-\omega_1^2x-\int_0^t{\rm d}t'\,\gamma(t-t')\ddot x(t')+(1/m_1)F^{\rm st}(t)$$

(where F^{st} stands for the radiation field force), which can be written as

$$x(t) = x_0\chi_1(t) + \dot{x}_0\chi_0(t) + \int_0^t \mathrm{d}s\, \chi_2(t-s)F^{st}(s)/m_1 \tag{6.2a}$$

$$\dot{x}(t) = -x_0\omega_1^2\chi_2(t) + \dot{x}_0\chi_1(t) + \int_0^t \dot{\chi}_2(t-s)(F^{st}(s)/m_1)\,\mathrm{d}s \tag{6.2b}$$

is also a Gaussian stochastic process whose mean is the solution without field, i.e. expressions (3.1*a*) and (3.1*b*), and whose covariance matrix is given by

$$C_{xx} \equiv \langle x^2(t)\rangle - \langle x(t)\rangle^2 = \frac{k_B T}{m_1}\left(\frac{1-\chi_1^2(t)}{\omega_1^2} - \chi_2^2(t)\right) \tag{6.3a}$$

$$C_{\dot{x}\dot{x}} \equiv \langle \dot{x}^2(t)\rangle - \langle \dot{x}(t)\rangle^2 = \frac{k_B T}{m_1}(1 - \omega_1^2\chi_2^2(t) - \dot{\chi}_2^2(t)) \tag{6.3b}$$

$$C_{x\dot{x}} \equiv \langle x(t)\dot{x}(t)\rangle - \langle x(t)\rangle\langle \dot{x}(t)\rangle = \frac{k_B T}{m_1}\chi_2(t)[\chi_1(t) - \dot{\chi}_2(t)]. \tag{6.3c}$$

As concerns the stationary probability density, the general result is obtained in Blanco *et al* (1986, § VC1). We only want to call attention to the fact that the solution is not in general given by the Maxwell-Boltzmann (MB) distribution law unless the condition $\varepsilon \ll 1$ is fulfilled, which corresponds to radii not too small.

On the other hand, we shall see now that the results of Blanco *et al* (1986) concerning the perturbation of the solutions caused by the radiation field are also satisfied for the class of charge distribution considered in the preceding sections. The relative deviations for the position, the velocity and the acceleration are calculated in Blanco *et al* (1986):

$$\Delta x \sim \frac{m\omega_0^2 C_{xx}}{k_B T(2+\varepsilon)} = \frac{1+\varepsilon}{2+\varepsilon}\omega_0^2\left(\frac{1-\chi_1^2}{\omega_1^2} - \chi_2^2\right) \tag{6.4a}$$

$$\Delta v \sim \frac{m C_{\dot{x}\dot{x}}}{k_B T(2+\varepsilon)} = \frac{1+\varepsilon}{2+\varepsilon}(1 - \omega_1^2\chi_2^2 - \dot{\chi}_2^2) \tag{6.4b}$$

$$\Delta a \sim \frac{m C_{aa}}{\omega_0^2 k_B T(2+\varepsilon)} = \frac{1+\varepsilon}{\omega_0^2(2+\varepsilon)}(\dot{\gamma}(0) + \omega_1^2(1-\dot{\chi}_2^2) - \ddot{\chi}_2^2) \tag{6.4c}$$

where $C_{aa} = \langle \ddot{x}^2(t)\rangle - \langle \ddot{x}(t)\rangle^2$ and only trajectories with energy of the order of the mean energy, $\mathscr{E} \sim \langle \mathscr{E}\rangle = k_B T(1+\varepsilon/2)$, have been considered. In the last expression, use has been made of the important fluctuation-dissipation property

$$B(t) \equiv \langle F^{st}(t+t_1)F^{st}(t_1)\rangle = k_B T m_1 \dot{\gamma}(|t|) \tag{6.5}$$

valid only for the RJ spectrum.

Now, from equations (5.28), (5.38*a*, *b*, *c*) and (5.44*a*, *b*, *c*) we obtain, for $5\omega_0\tau_e \ll \omega_0 t \ll \xi^{-1} \simeq 2/\omega_0\tau_0$,

$$\Delta x \sim \frac{\varepsilon}{(1+\varepsilon)(2+\varepsilon)}\sin^2(\omega_0\nu t) \tag{6.6a}$$

$$\Delta v \sim \frac{\varepsilon}{1+\varepsilon}\left(1 - \frac{1}{2+\varepsilon}\sin^2(\omega_0\nu t)\right) \tag{6.6b}$$

$$\Delta a \sim \frac{(1+\varepsilon)\dot{\gamma}(0)}{(2+\varepsilon)\omega_0^2} + \varepsilon\left(1 + \frac{\sin^2(\omega_0\nu t)}{(1+\varepsilon)(2+\varepsilon)}\right). \tag{6.6c}$$

We immediately see that the two first expressions coincide with those obtained for the Yukawa distribution. For the last one, we must estimate $\dot{\gamma}(0)$. From (2.4a) and (2.8) we may write

$$\dot{\gamma}(0) = 16(1+\varepsilon)\pi^2\tau_0 \int_0^\infty \omega^2\hat{\rho}^2(\omega)\,\mathrm{d}\omega. \tag{6.7}$$

If we admit that $\hat{\rho}^2$ is approximately constant until ω reaches the cut-off value $\omega_c \sim \tau_e^{-1}$ and then vanishes, we may estimate the integral appearing in (6.7) by $\frac{1}{3}\hat{\rho}^2(0)\omega_c^3$ and then, as $\hat{\rho}(0) = (2\pi)^{-3/2}$,

$$\dot{\gamma}(0) \sim (2/3\pi)(1+\varepsilon)\tau_0/\tau_e^3. \tag{6.8}$$

Consequently, in (6.6c) we may write for the first term on the RHS

$$\frac{(1+\varepsilon)\dot{\gamma}(0)}{(2+\varepsilon)\omega_0^2} \sim \frac{2}{3\pi}\frac{(1+\varepsilon)^2}{(2+\varepsilon)}\frac{\tau_0}{\omega_0^2\tau_e^3} < \frac{\varepsilon}{(\omega_0\tau_e)^2} \tag{6.9}$$

where (5.4) has been used.

Obviously, this term predominates in (6.6c), and then

$$\Delta a \sim \varepsilon/(\omega_0\tau_e)^2. \tag{6.10}$$

This expression also coincides with that obtained for the Yukawa case.

Therefore, the discussion appearing in Blanco *et al* (1986) on the perturbative effect of the radiation field can be extended to the class of charge distributions studied in the present paper.

The theme of the energy interchange will also present the same characteristics as the Yukawa case. However, the study is somewhat more involved. We now seek to clarify this point with a frequency analysis of the energy interchange.

The power absorbed by the charge in the frequency range between ω and $\omega + \mathrm{d}\omega$ is given by

$$P_a(\omega, \omega+\mathrm{d}\omega) \equiv I_a(\omega)\,\mathrm{d}\omega = \frac{\mathrm{d}\omega}{\pi}\int_{-\infty}^{\infty} \mathrm{d}u\langle v(t)F^{\mathrm{st}}(t+u)\rangle_{\mathrm{st}} \cos\omega u \tag{6.11}$$

where the average has to be calculated in the stationary state. A derivation of that expression can be found in Blanco and Pesquera (1986).

Now, by using (6.2b) for $t \to \infty$, the relations (6.5), (2.4a), (3.5c) and the properties of the Laplace transform, we get after some algebra

$$I_a(\omega) = (2/\pi)k_B T\omega^4[\mathrm{Im}\,\tilde{\gamma}(-\mathrm{i}\omega)]^2|\tilde{\chi}_2(-\mathrm{i}\omega)|^2. \tag{6.12}$$

In order to understand the shape of $I_a(\omega)$ as a function of ω, we may consider two regions: $\omega \ll \tau_e^{-1}$ and $\omega \gtrsim \tau_e^{-1}$. In the former we expand $\tilde{\gamma}$ in a power series of $\omega\tau_e$ (see equation (5.20)). The first non-vanishing contribution to $I_a(\omega)$ gives

$$I_a(\omega) \simeq \frac{2}{\pi}k_B T\frac{\omega^6\tau_0^2}{(\omega_0^2-\omega^2)^2+\omega^6\tau_0^2} \qquad \omega \ll \tau_e^{-1} \tag{6.13a}$$

which is exactly the expression obtained when the Abraham–Lorentz (AL) model is used for the charge.

In the case $\omega \gtrsim \tau_e$, condition (G) lets us write

$$|\omega^2(1+\tilde{\gamma})| > \omega^2(1-\varepsilon) \gg \omega_0^2(1+\varepsilon)\tfrac{1}{3}$$

and we may approximate I_a by its value corresponding to the free particle, $\omega_0=0$,

$$I_a(\omega) \simeq \frac{2}{\pi} k_B T \frac{(\text{Im}\,\tilde{\gamma}(-i\omega))^2}{|1+\tilde{\gamma}(-i\omega)|^2} \qquad \omega \gtrsim \tau_e^{-1}. \tag{6.13b}$$

Now, the total contribution of small frequencies ($\omega \ll \tau_e^{-1}$) is well known from the AL model to be of order

$$P_a^{(L)} \sim (k_B T)\omega_0^2 \tau_0. \tag{6.14}$$

Moreover, the free-particle expression is negligible at small frequencies and coincides with the oscillator expression at high frequencies. As we shall see now, the total power for the free case is in general much bigger than the contribution of the peak around ω_0 in expression (6.12). Consequently, the total absorbed power for the oscillator will be given in the first approximation by the free-particle value ($\omega_0=0$). It is given by

$$\frac{P_a}{\langle \mathscr{E} \rangle} = \int_0^\infty d\omega \frac{2}{\pi(1+\frac{1}{2}\varepsilon)} \frac{[\text{Im}\,\tilde{\gamma}(-i\omega)]^2}{|1+\tilde{\gamma}(-i\omega)|^2}. \tag{6.15}$$

To estimate this quantity, note that,

$$|1+\tilde{\gamma}| < 1+\varepsilon$$

and then

$$\frac{P_a}{\langle \mathscr{E} \rangle} > \frac{4}{\pi(2+\varepsilon)(1+\varepsilon)^2} \int_0^\infty d\omega [\text{Im}\,\tilde{\gamma}(-i\omega)]^2$$
$$= \frac{4(8\pi^3)^2 \tau_0^2}{\pi(2+\varepsilon)} \int_0^\infty \omega^2 \hat{\rho}^4(\omega)\, d\omega. \tag{6.16}$$

The same arguments leading to (6.8) enable us to estimate this expression as

$$\frac{P_a}{\langle \mathscr{E} \rangle} \gtrsim \frac{4}{3\pi(2+\varepsilon)} \frac{\tau_0^2}{\tau_e^3}. \tag{6.17}$$

In one period, ω_0^{-1}, the energy interchanged is

$$\frac{\Delta \mathscr{E}}{\langle \mathscr{E} \rangle} \gtrsim \frac{4}{3\pi(2+\varepsilon)} \frac{(\tau_0/\tau_e)^2}{\omega_0 \tau_e}. \tag{6.18}$$

Although both the numerator and the denominator are small, only for high enough frequencies will expression (6.18) not be large. Then we again obtain, as in the Yukawa case, a strong interaction between the radiation field and the charge.

7. Conclusions

We have presented a rather exhaustive analysis of the motion of a rigid spherically symmetric extended charge in the presence of a linear force field in the non-relativistic approximation. A short analysis of its interaction with RJ radiation also appears. However, not all kinds of charge distributions have been treated. Some restrictions have been necessary so as to be able to specify the solutions more or less explicitly. For instance, ρ has a definite sign and $\tilde{\gamma}$ must have a strictly negative abscissa of convergence; the radius is too large and the frequency too small so as to have $0<\varepsilon<0.5$

and $\omega_o\tau_e \ll 1$, respectively, in § 5, etc. The need for these restrictions appears clearly in the analysis developed in the various sections.

We have obtained the general form of solutions including non-radiating oscillations but not runaways. For radii not too small ($0<\varepsilon<0.5$) and frequencies not too large ($\omega_0\tau_e \ll 1$) we obtain that the motion is similar to the one corresponding to the AL model, the several parameters coinciding in the first order of a $\omega_0\tau_e$ expansion.

On the other hand, for this kind of charge distribution, the interaction with the RJ radiation is small as concerns the phase-space trajectory only if the electromagnetic mass is negligible compared with the mechanical one. However, this interaction will be strong in general if higher derivatives of the electron position are considered, a fact that is due to the effect of the high frequencies.

As a matter of fact we have seen that the behaviour displayed by a Yukawa charge distribution (see Blanco *et al* 1986) also occurs in the case of the wide class of charge distributions considered in this paper. On the other hand, as the analysis presented shows, it could be said that we have characterised the kind of charge distributions for which the motion displays a behaviour similar to the motion ruled by the AL model, which then becomes a good approximation to the extended model for such distributions. Moreover, the analysis seems to indicate that, most probably, for other charge distributions another behaviour is to be expected that is not well represented by the AL model. For instance, the possibility of having non-radiating solutions is undoubtedly not shared by this point model. However, not all the distributions excluded from this paper will necessarily display a 'new' type of behaviour (see Blanco *et al* 1986) and the search of structures having such 'strange' solutions has in our opinion great interest.

Appendix 1

To obtain (4.28) we first prove the following.

Lemma. Let f be an absolutely integrable real-valued function with second derivative and satisfying

$$x>0 \Rightarrow -f(-x)=f(x)>0. \tag{A1.1}$$

Let $a>0$ be a zero of f of multiplicity 2. Then

$$\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathrm{VP}\int_{-\infty}^{\infty}\frac{f(x)}{x-\alpha}\,\mathrm{d}x\right)\bigg|_{\alpha=a}>0. \tag{A1.2}$$

Proof. We first define the function

$$\varphi(x,\alpha)=\begin{cases}\dfrac{f(x)-f(\alpha)-f'(\alpha)(x-\alpha)}{(x-\alpha)^2} & x\neq\alpha \qquad \text{(A1.3}a\text{)}\\[2ex] \lim\limits_{x\to\alpha}\dfrac{f(x)-f(\alpha)-f'(\alpha)(x-\alpha)}{(x-\alpha)^2} & x=\alpha. \qquad \text{(A1.3}b\text{)}\end{cases}$$

The limit exists because of the properties of f and so φ is continuous at $x=\alpha$ too. The following transformations are immediate.

For any $A>0$ we may write

$$K(\alpha)\equiv \mathrm{VP}\int_{-\infty}^{\infty} \mathrm{d}x\,\frac{f(x)}{x-\alpha}=\mathrm{VP}\int_{-\infty}^{\infty}\mathrm{d}t\,\frac{f(t+\alpha)}{t}$$
$$=\mathrm{VP}\int_{-A}^{A}\mathrm{d}t\left(\frac{f(\alpha)}{t}+f'(\alpha)+t\varphi(t+\alpha,\alpha)\right)+\int_{|t|>A}\mathrm{d}t\,\frac{f(t+\alpha)}{t}$$
$$=2Af'(\alpha)+\int_{-A}^{A}\mathrm{d}t\,t\varphi(t+\alpha,\alpha)+\int_{|t|>A}\frac{f(t+\alpha)}{t}\,\mathrm{d}t. \tag{A1.4}$$

Deriving

$$K'(a)\equiv K'(\alpha)|_{\alpha=a}$$
$$=2Af''(a)+\int_{|t|>A}\frac{f'(t+A)}{t}\,\mathrm{d}t+\int_{-A}^{A}\mathrm{d}t\,t\left.\frac{\partial\varphi(x,\alpha)}{\partial\alpha}\right|_{\substack{x=t+a\\ \alpha=a}}$$
$$+\int_{-A}^{A}\mathrm{d}t\,t\,\frac{\mathrm{d}\varphi(t+a,a)}{\mathrm{d}t}$$
$$=\int_{|t|>A}\frac{f'(t+a)}{t}\,\mathrm{d}t+2Af''(a)+\int_{-A}^{A}\mathrm{d}t\,t\left.\frac{\partial\varphi(x,\alpha)}{\partial\alpha}\right|_{\substack{x=t+a\\ \alpha=a}}$$
$$+A[\varphi(A+a,a)-\varphi(-A+a,a)]-\int_{-A}^{A}\mathrm{d}t\,\varphi(t+a,a). \tag{A1.5}$$

From (A1.3a, b) we obtain

$$t\left.\frac{\partial\varphi(x,\alpha)}{\partial\alpha}\right|_{x=t+\alpha}=-f''(\alpha)+2\varphi(t+\alpha,\alpha) \tag{A1.6}$$

whence (A1.5) yields

$$K'(a)=\int_{|t|>A}\frac{f'(t+a)}{t}\,\mathrm{d}t+A[\varphi(A+a,a)-\varphi(-A+a,a)]+\int_{-A}^{A}\varphi(t+a,a)\,\mathrm{d}t. \tag{A1.7}$$

Since a is a double zero of f, we have

$$f(a)=f'(a)=0 \tag{A1.8}$$

and then

$$A\varphi(\pm A+a,a)=f(\pm A+a)/A. \tag{A1.9}$$

Expression (A1.7) is valid for all A, and in particular we may perform the limit $A\to\infty$. The two first terms in (A1.7) vanish, the second one because of (A1.9) and both due to the integrability of f. Consequently

$$K'(a)=\int_{-\infty}^{\infty}\mathrm{d}t\,\varphi(t+a,a)=\int_{-\infty}^{\infty}\varphi(t,a)\,\mathrm{d}t. \tag{A1.10}$$

Now, in virtue of (A1.1) we obtain from (A1.3a, b)

$$\varphi(t,a)\equiv f(t)/(t-a)^2>0 \qquad t>0 \tag{A1.11}$$

$$\varphi(-t,a)=\frac{f(-t)}{(t+a)^2}=-\frac{f(t)}{(t+a)^2}>-\frac{f(t)}{(t-a)^2} \qquad t>0 \tag{A1.12}$$

and then

$$t>0 \Rightarrow \varphi(-t, a)+\varphi(t, a)>0 \tag{A1.13}$$

whence, finally

$$K'(a)=\int_{-\infty}^{\infty} \varphi(t, a)\, \mathrm{d}t=\int_{0}^{\infty} [\varphi(t, a)+\varphi(-t, a)]\, \mathrm{d}t>0 \tag{A1.14}$$

which proves the lemma.

Going back to the point in which we are interested, i.e. the proof of (4.28), in (4.22) we see that

$$\operatorname{Re} \tilde{\gamma}(\mathrm{i}\mu_0)=\frac{16\pi^2 e^2}{3m_1 c^3} \mathrm{VP} \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\omega}{\omega-\mu_0} \hat{\rho}^2(|\omega|). \tag{A1.15}$$

Let us define now, as the f function of the lemma,

$$f(x)=\frac{16\pi^2 e^2}{3m_1 c^3} x\hat{\rho}^2(|x|). \tag{A1.16}$$

Now, on the one hand

$$|\hat{\rho}(|\omega|)| \leqslant \hat{\rho}(0)=1/(2\pi)^{3/2} \tag{A1.17}$$

and, on the other hand, the existence of $\dot{\gamma}$ comes from the existence of the integral (see Blanco *et al* 1986)

$$\int_{0}^{\infty} \omega^2 \hat{\rho}^2(\omega)\, \mathrm{d}\omega. \tag{A1.18}$$

But this fact demands that $\hat{\rho}^2$ converges to zero faster than ω^{-3} as $\omega \rightarrow \infty$, and consequently $\omega\hat{\rho}^2$ goes to zero faster than ω^{-2} whence we obtain the absolute integrability of $f(x)$.

As concerns the second derivative of (A1.16), it is easy to see from (2.4*a*) that

$$2\pi f(\omega)=\int_{0}^{\infty} \mathrm{d}t\, \gamma(t) \sin \omega t \tag{A1.19}$$

whence it follows that

$$|f''(\omega)| \leqslant \frac{1}{2\pi} \int_{0}^{\infty} \mathrm{d}t\, \gamma(t) t^2. \tag{A1.20}$$

Now, by virtue of (4.3) the last integral is finite, which proves the existence of the second derivative of f. Relations (A1.1) are trivial. Finally, since μ_0 must satisfy equation (4.23*a*), it is clear that it is a double zero of f. Consequently the lemma applies and it results that (note $m_1>0$)

$$\frac{\mathrm{d}}{\mathrm{d}\mu_0} \operatorname{Re} \tilde{\gamma}(\mathrm{i}\mu_0)=\left(\frac{\mathrm{d}}{\mathrm{d}\alpha} \int_{-\infty}^{\infty} \mathrm{d}x \frac{f(x)}{x-\alpha}\right)\Bigg|_{\alpha=\mu_0}>0 \tag{A1.21}$$

which proves (4.28).

Appendix 2

We are going to obtain the bounds for $R_i(t)$ given by (5.42).

Let us remember that along C_2, $\operatorname{Re} p = -1/5\tau_e$, and then

$$|\tilde{\gamma}(z)| \leqslant \tilde{\gamma}(-1/5\tau_e) = \int_0^{2\tau_e} \gamma(t) \exp(t/5\tau_e)\, dt < \varepsilon\, e^{2/5} \tag{A2.1}$$

whence we may write

$$|R_i(t)| \leqslant \frac{(1+e^{2/5})}{2\pi} \varepsilon \exp(-t/5\tau_e) \int_{-\infty}^{\infty} d\lambda \frac{A_i(z)}{|\omega_0^2(1+\varepsilon) + z^2(1+\tilde{\gamma})||\omega_0^2 + z^2|} \tag{A2.2}$$

where

$$\lambda = \operatorname{Im} z \tag{A2.3}$$

$$A_0 = \omega_0^2 \tag{A2.4a}$$

$$A_1 = \omega_0^2 |z| \tag{A2.4b}$$

$$A_2 = |z|^2/(1+\varepsilon). \tag{A2.4c}$$

Making now the change

$$\lambda \tau_e = x \tag{A2.5}$$

and calling

$$\delta = \omega_0 \tau_e \tag{A2.6}$$

(A2.2) may be written

$$|R_i(t)| \leqslant \frac{(1+e^{2/5})\varepsilon}{2\pi} \times \tau_e \exp(-t/5\tau_e) \int_{-\infty}^{\infty} dx \frac{\bar{A}_i}{|\delta^2(1+\varepsilon) + (-\frac{1}{5}+ix)^2(1+\tilde{\gamma})||\delta^2 + (-\frac{1}{5}+ix)^2|} \tag{A2.7}$$

with

$$\bar{A}_0 = \omega_0^2 \tau_e^2 \tag{A2.8a}$$

$$\bar{A}_1 = \omega_0^2 \tau_e |-\tfrac{1}{5}+ix| \tag{A2.8b}$$

$$\bar{A}_2 = \frac{1}{1+\varepsilon} |-\tfrac{1}{5}+ix|^2. \tag{A2.8c}$$

Let us consider the quantity

$$Q = \frac{(\tilde{\gamma}-\varepsilon)(-\frac{1}{5}+ix)^2}{[\delta^2 + (-\frac{1}{5}+ix)^2](1+\varepsilon)} \tag{A2.9}$$

which allows us to write

$$\delta^2(1+\varepsilon) + (-\tfrac{1}{5}+ix)^2(1+\tilde{\gamma}) = [\delta^2 + (-\tfrac{1}{5}+ix)^2](1+\varepsilon)(1+Q). \tag{A2.10}$$

A bound can be found for Q. Firstly we may put

$$|Q| \leqslant \frac{(1+e^{2/5})}{1+\varepsilon} \varepsilon [y(x)]^{1/2} \tag{A2.11}$$

with

$$y(x)=\frac{(\frac{1}{25}+x^2)^2}{(\delta^2+\frac{1}{25}-x^2)^2+\frac{4}{25}x^2} \tag{A2.12}$$

where use has been made of (A2.1). Now, the maximum of $y(x)$ is easily found to be at

$$x_0^2=\delta^2+\tfrac{3}{25} \tag{A2.13}$$

and to have the value

$$y(x_0)=y_{\max}=1+\tfrac{25}{4}\delta^2. \tag{A2.14}$$

Consequently we may write

$$|Q|\leqslant\frac{(1+e^{2/5})}{1+\varepsilon}\varepsilon(1+\tfrac{25}{4}\delta^2)^{1/2}\equiv\eta<0.9 \tag{A2.15}$$

whence we obtain

$$|1+Q|>1-\eta>0.1. \tag{A2.16}$$

From (A2.10) and (A2.16), (A2.7) becomes

$$|R_i(t)|\leqslant\frac{(1+e^{2/5})\varepsilon\tau_e}{\pi(1+\varepsilon)(1-\eta)}\exp(-t/5\tau_e)\int_0^\infty dx\frac{\bar{A}_i}{|\delta^2+(-\frac{1}{5}+ix)^2|^2}. \tag{A2.17}$$

A procedure similar to the one used above will allow us to simplify this expression. Let us introduce

$$\bar{Q}=\delta^2/(-\tfrac{1}{5}+ix)^2. \tag{A2.18}$$

Now

$$\delta^2+(-\tfrac{1}{5}+ix)^2=(-\tfrac{1}{5}+ix)^2(1+\bar{Q}). \tag{A2.19}$$

For $\bar{Q}$, it is easy to find

$$|\bar{Q}|=\delta^2/(\tfrac{1}{25}+x^2)<25\delta^2=\eta'\ll 1 \tag{A2.20}$$

by which

$$|1+\bar{Q}|>1-\eta'=O(1) \tag{A2.21}$$

and then

$$|R_i(t)|\leqslant\frac{(1+e^{2/5})\varepsilon\tau_e}{\pi(1+\varepsilon)(1-\eta)(1-\eta')}\exp(-t/5\tau_e)\int_0^\infty dx\frac{\bar{A}_i}{(\frac{1}{25}+x^2)^2}. \tag{A2.22}$$

Finally, taking into account that

$$\int_0^\infty dx\frac{1}{(\frac{1}{25}+x^2)^2}=\tfrac{125}{4}\pi \tag{A2.23a}$$

$$\int_0^\infty dx\frac{1}{(\frac{1}{25}+x^2)^{3/2}}<125\int_0^\infty dx\frac{1}{(1+25x^2)}=\tfrac{25}{2}\pi \tag{A2.23b}$$

$$\int_0^\infty dx\frac{1}{(\frac{1}{25}+x^2)}=\tfrac{5}{2}\pi \tag{A2.23c}$$

and replacing η and η' by their values we obtain expressions (5.44*a*, *b*, *c*).

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